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Strong pseudoconvexity in Banach spaces

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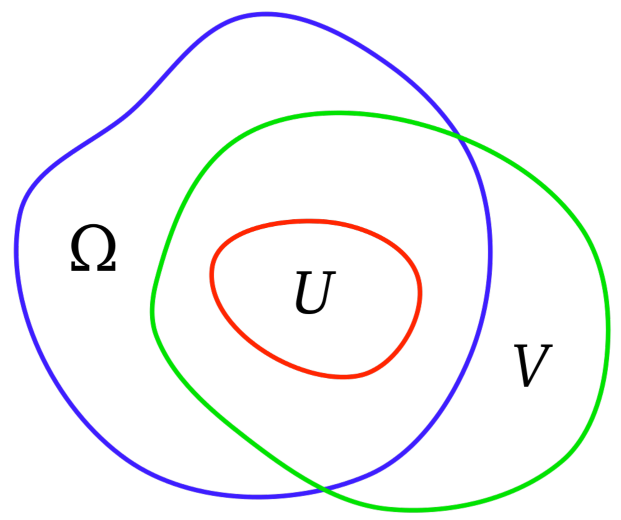
1 Preliminaries

In $\mathbb{C}$, every open domain admits an analytic function that cannot be extended across its boundary. In several variables, Hartogs' domains exhibit a break with this behavior, however domains of holomorphy capture such commonality with one variable.



In separable Banach spaces X with the BAP, the global property of being a domain of holomorphy is described in local analytic/geometric terms by pseudoconvexity [Levi, 1910-1911]: Given r a defining function of the boundary, the Levi condition holds at each boundary point a:

$$D'D''r(a)(b,b) \geq 0 \text{ for all } b \in \mathbb{C}^n \text{ such that } D'r(a)b = 0. \tag{1}$$

If there is strict inequality for $b \neq 0$, the domain is called strictly pseudoconvex. In finite dimension, the latter are locally biholomorphic to strongly convex sets.

In the 1940s, Oka and Lelong characterized and extended pseudoconvexity as convexity w. r. t. plurisubharmonic functions: upper semicontinuous functions $f : U \to [-\infty, \infty)$, s. t. for each $a \in U$ and $b \in X$ with $a + \mathbb{D} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

If $f \in C^2(U, \mathbb{R})$, it is plurisubharmonic iff for all $a \in U$ and $b \in X$, $D'D''f(a)(b,b) \geq 0$; and it is strictly plurisubharmonic if there is strict inequality for $b \neq 0$. In fact, a domain U in $\mathbb{C}^n$ with C^2 boundary is strictly pseudoconvex iff there is a strictly p. s. h. function defining the boundary.

While an open domain is convex iff $-\log d_U$ is a convex function, in turn U is pseudoconvex iff $-\log d_U$ is plurisubharmonic. In finite dimension, pseudoconvexity is equivalent to the existence of a plurisubharmonic exhaustion function of the domain.

2 Novelties on strong pseudoconvexity

Theorem (O-C, '16). *If U is an open domain in $\mathbb{C}^n$ with C^2 boundary, U is strictly pseudoconvex iff there exist V a neighborhood of $\bar{U}$, $\rho \in C^2(V)$ a defining function of ∂U, and $\varphi \in C^\infty(U)$ strictly positive such that $\inf_{a \in U} \varphi(a)|\rho(a)| > 0$ and,*

$$D'D''(-\log|\rho|)(a)(b,b) \geq \varphi(a)\|b\|^2 \text{ for all } a \in U \text{ and } b \in \mathbb{C}^n. \tag{2}$$

A generalization of strict p. s. h.: an upper semicontinuous $g : U \subset X \to [-\infty, \infty)$ is strictly plurisubharmonic on average if there exists $\varphi \in C^\infty(U)$ positive such that for all $a \in U$ and $b \in \mathbb{C}^n$ of small norm (size depending on a),

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta. \tag{3}$$

If g is real-valued and we can find φ constant, g is uniformly plurisubharmonic.

In turn, a connected domain U in $\mathbb{C}^n$ is strictly pseudoconvex on average if there exists $\rho \not\equiv -\infty$ strictly p. s. h. on average in a neighborhood V of $\bar{U}$ such that $U = \{z \in V : \rho(z) < 0\}$. If U connected is in a Banach space, it is strongly pseudoconvex if it is so on average in each finite-dimensional subspace; and if there exists ρ as before and uniformly plurisubharmonic, U is called uniformly pseudoconvex.

3 Examples and counterexamples

Every open and convex domain is pseudoconvex, but not necessarily strongly pseudoconvex; e. g. polydisks are convex though not strongly pseudoconvex. Furthermore, there is a pseudoconvex domain smoothly bounded, that is strongly pseudoconvex except at one boundary point [Sibony, '87].

To find examples of Banach spaces whose unit ball is strongly pseudoconvex, we considered a number of complex analogues of uniform convexity, until the one below.

Theorem (O-C, '14). *If X is a 2-uniformly PL-convex Banach space then B_X is uniformly pseudoconvex.*

For $p \in [1,2]$, $B_{L_p(\Sigma,\Omega,\mu)}$ is 2-uniformly PL-convex [Davis, Garling, Tomczak-Jaegermann, '84]. Meanwhile, for $2 < p \leq \infty$ and $n \geq 2$, the ball of ℓ_p^n lacks strong pseudoconvexity, and so do the balls of ℓ_p and L_p for $p > 2$. The same prop. fails for $B_{C(K)}$, for K compact and Hausdorff with $|K| \geq 2$ (it contains the polydisk $B_{\ell_\infty^2}$).